PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Gulf Stream

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The Gulf Stream

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The dynamics of the Gulf Stream in the meander region from Cape Hatteras to the Grand Banks will be examined in the light of new theoretical and observational evidence. The theory of topographic meandering and baroclinic instability will be discussed, and a time-dependent thin-jet equation derived. Simultaneous observations of the path and structure of the stream, including long records of strong deep velocity fluctuations, will be analysed. Two weeks of repetitive tracking over 2° of longitude revealed a secular change of the surface path, the advection of a recognizable feature, and a short wave motion (wavelength comparable to the Stream width). The vorticity and the vorticity balance of the thin jet equation are computed from the observations; transient contributions to the vorticity are found to be comparable to the quasi-steady terms. A general quasigeostrophic model is outlined in which the path equation for a broad baroclinic current is shown to be identical to the linearized thin jet equation. A solution to this equation is presented which exhibits an eddy production mechanism.

1. The Gulf Stream

The Gulf Stream is the major current system involved in the circulation of the North Atlantic subtropical gyre. The current forms in the Caribbean, flows through the Straits of Florida, and thence northward over the shallow Blake plateau to Cape Hatteras, intensifying in transport. There it leaves the shelf and meanders in deep water eastward to the Grand Banks. Beyond the Banks information is scanty, but the Stream becomes less well defined, filaments and transfers its transport to the slow and broad southward drift of the gyre.

Here I shall discuss the dynamics of the Stream in the meander region between Cape Hatteras and the Grand Banks, which is also a region of eddy formation and shedding. I shall first briefly summarize the observational data base and mention previous theoretical studies, then report some results from an experiment which involved the simultaneous measurement of several stream variables, outline the structure of a new time-dependent mathematical model, and finally describe the results of some theoretical studies recently completed or in progress at Harvard University. Complete arguments and qualifications cannot of course be presented here but the appropriate research articles will be cited for details.

This region of the sea is relatively well explored, and although observations of the meanders and eddies are indeed insufficient for an unambiguous qualitative and quantitative definition of these phenomena, they do provide a relatively good testing ground for theoretical ideas about ocean currents. To test the ideas, it is of course necessary to cast them in a form specifically applicable to the phenomena, and to develop a feedback between theoretical studies and ongoing observational experiments. To achieve a degree of quantitative comparison, the ocean current theorist must exploit the capabilities of experimental techniques with a full awareness of their limitations. A direct comparison between a theoretical prediction and an observed fact is usually impossible. In the present case theoretical ideas are subject to test by the use of theoretical considerations to interrelate different classes of observational data (hydrographic data, direct velocity measurements, and data on the location of the Stream).

The meanders and eddies, of obvious dynamical interest *per se*, also present an important problem which must be resolved in the theory of the general ocean circulation. It is my opinion

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that the dynamics of relevant intermediate scale processes must be understood in a reasonably detailed way in order to allow for their physically correct parameterization into the large-scale mean flow.

2. Meanders and eddies

A considerable amount of observational information on the location (or path) of the Stream in this region exists. This information has not been obtained primarily by direct observation of

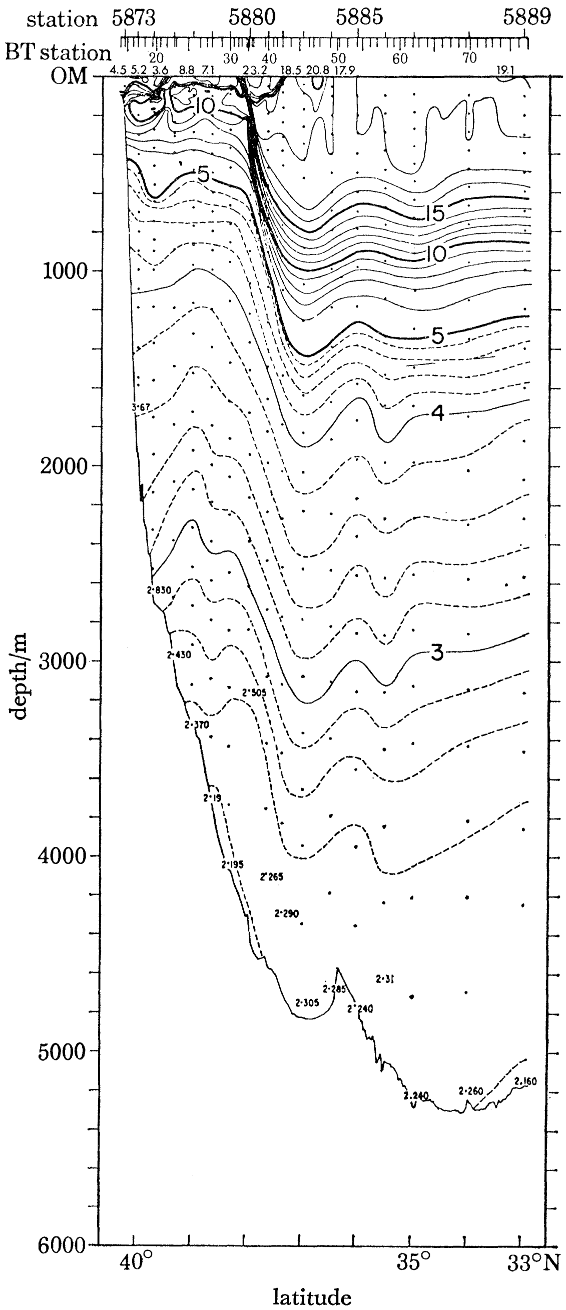

Figure 1. A temperature section across the Gulf Stream taken on 9 April 1960, at 68° 30′ W longitude. After Fuglister (1963) Atlantis Cruise 255, Section I.

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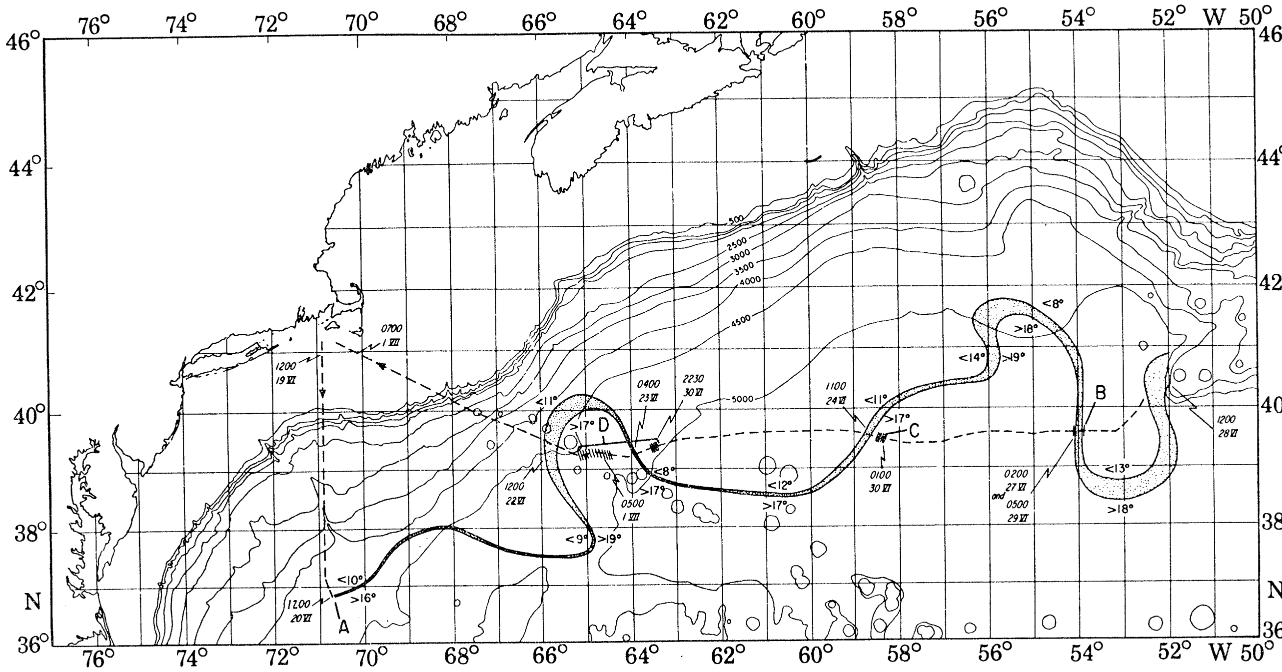

FIGURE 2. The path of the Gulf Stream in late June 1964, inferred from temperatures at 200 m depth. After Fuglister & Voorhis (1965, Figure 2).

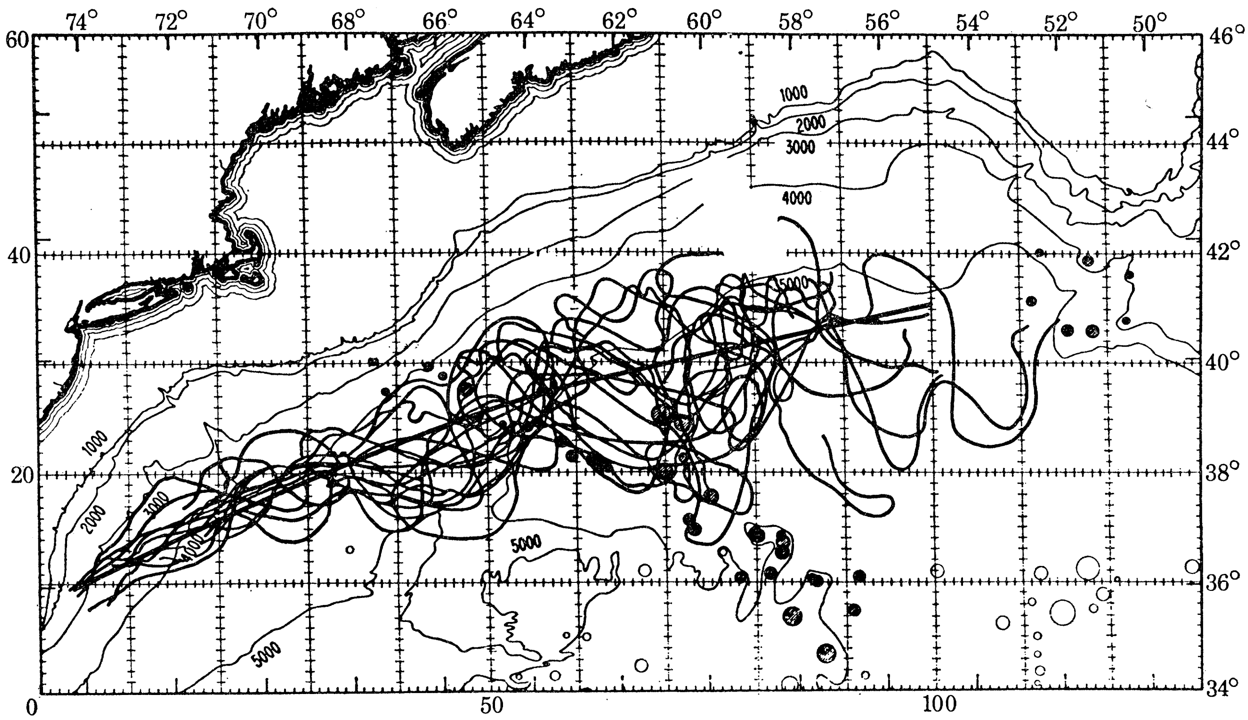

FIGURE 3. Composite of all Gulf Stream position data available for 20 years before 1966. After Niiler & Robinson (1967, Figure 15).

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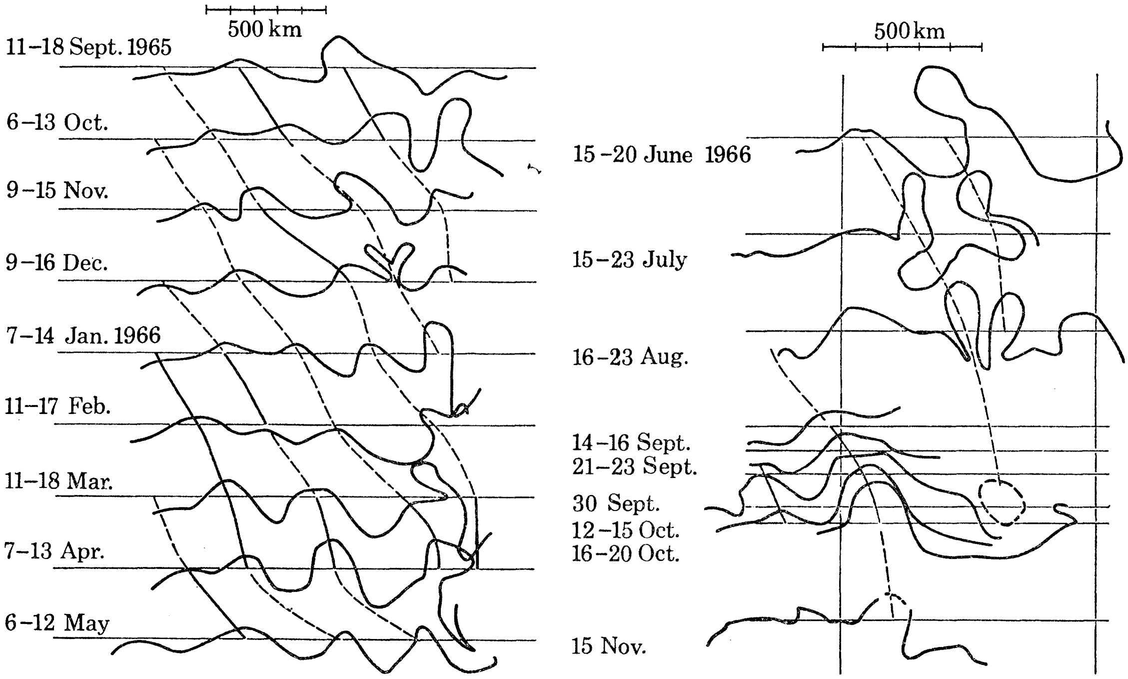

FIGURE 4. Phase propagation interpretation of successive Stream positions. After Hansen (1970, Figure 3).

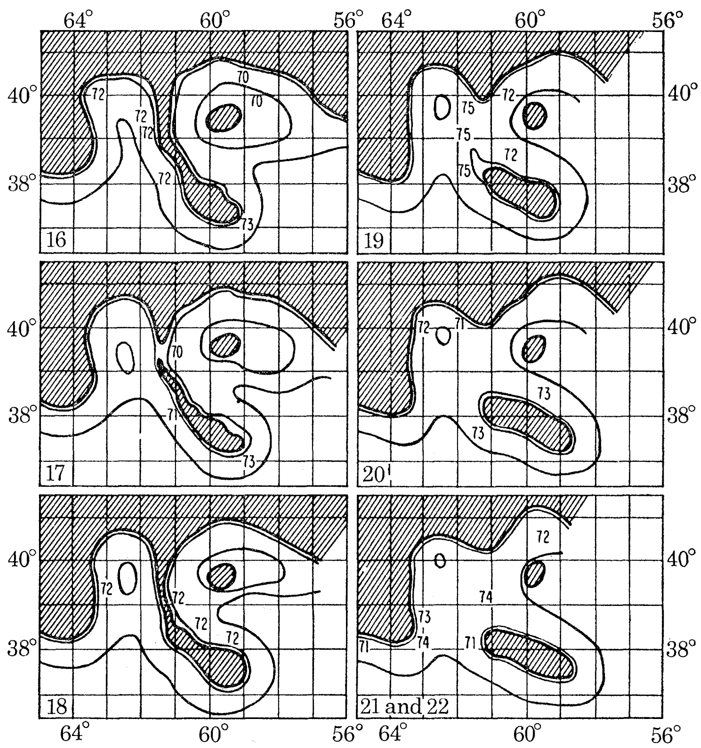

FIGURE 5. The formation of a detached eddy from an elongated meander from 16 to 22 June, 1950. Temperature in degrees Farenheit. After Fuglister & Worthington (1951, Figure 6b).

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velocity, but by identification of the axis of the Stream with the 15° isotherm at a subsurface depth of 200 m (Fuglister & Voorhis 1965). The current is geostrophic, baroclinic, narrow and swift. Thus (see figure 1) the pattern of isotherms in a vertical plane normal to the current exhibits a densely packed, highly sloped region in the vicinity of the most intense current (since the vertical shear of the downstream flow is proportional to the cross-stream temperature gradient via thet hermal wind relationship (Stommel 1966, chapter 3)). The 15 °C temperature at 200 m (T_{15}) is indicative of this thermal feature; thus relatively rapid positioning of the Stream is possible. A typical Stream path is shown in figure 2. The T_{15} indicator is a better stream indicator than is the surface thermal ΔT_s (Hansen & Maul 1970) but the latter is useful and aircraft tracking of the current by infrared thermometry allows rapid coverage over large distances.

That T_{15} is indicative of the near surface velocity maximum has some direct empirical basis in the results of the tracking of near surface drogues, which move with the average speed of the current from the surface to, for example, 200 m depth. Only a few tracks have been obtained, but drogues launched over T_{15} have been observed to remain over T_{15} far downstream. Mr Fuglister has informed me that he has in one case followed such a drogue for 1500 km until he ceased tracking after 1500 km. The drogues move at nearly constant speed $\approx$ 200 cm/s which is believed to be the maximum speed in the current.

The statistical distribution of the Stream paths is displayed in figure 3; it may be simply characterized by an average axis, an envelope which widens eastward, and an average meander wavelength $\approx$ 300 km. The width of the Stream is less than 100 km. An extensive tracking programme was carried out by the U.S. Coast Guard Geodetic Survey from September 1966 to November 1967 (Hansen 1970). Initially a monthly sampling rate was employed, since the idea that the meanders were a quasi-steady phenomenon was then prevalent; but it was found necessary to employ a more frequent sampling of a shorter path segment during the latter portion of the experiment. Hansen has calculated 'phase-speeds' (mean value 7 cm/s) for the eastward propagation of meander features (figure 4), although discretion is required in such an interpretation of the path data.

That meanders can intensify, elongate, and pinch-off to the south as isolated cold-core eddies (see figure 5) has been known since the multiple ship survey of 1950 (Fuglister & Worthington 1951); this phenomenon has been further studied by Fuglister (1967). The shedding of warm core eddies to the north has recently been documented by Saunders (1970). That the two processes must differ considerably in their dynamics is indicated by a consideration of the geometry of the events taking into account the finite width of the Stream.

3. Previous theories

In the past, two theoretical approaches have been taken in the study of Gulf Stream meanders, first via hydrodynamic instability theory and secondly via the theory of steady topographic meandering. Neither theory is adequate to rationalize the phenomena.

In the baroclinic and barotropic instability theoretical approach, a steady current has been subject to infinitesimal perturbations of the form of travelling plane waves ($\exp i(kx-\sigma t)$, σ imaginary). The results, which have been summarized by Hansen (1970, Table 1), have been related to observations by comparing deduced phase speeds with phase speeds inferred from the data as exemplified in figure 4. The deduced temporal growth rates, after division by the deduced group velocity, have been compared with a spatial growth rate as indicated by the downstream spreading of the envelope of paths (figure 3). In the models studied deduced phase speeds and growth rates are generally considerably higher than observed. An exception is the recent study of Orlanski (1969) who obtained good agreement for phase speed (7 cm/s) for baroclinically unstable waves affected by bottom topography, although these waves amplify much too rapidly under the aforementioned criterion of comparison. In Orlanski's two-layer model the perturbations, but not the basic current, extend to the bottom; only cross-stream varieties of topography are included. I have studied the baroclinic instability of a current of arbitrary shape (in both vertical and cross-stream coordinates) flowing over a flat bottom in terms of a model described in § 8 below (equations (10) and (11)). The frequency σ I assume real, but the wavelength k, imaginary. Good results are obtained for phase speed, but again the downstream amplification is too rapid.

It seems apparent that a process of baroclinic instability may be inherent in the meandering of the Gulf Stream, but simple linearized models are obviously unsatisfactory for application to the finite-amplitude nonlinear meanders of the thin stream (figure 2). Comparison of calculated growth rates with the spread of the path envelope is of questionable significance. Rapid amplification may be a correct result, simply indicative of a very limited region of applicability of linearized analysis. In such a case the meaning of the comparison of observed and calculated phase speeds is moot. Furthermore, Dr Gadgil and I have shown (Robinson & Gadgil 1970) that in the meander region of the Stream retrograde and oscillatory wave propagation must be expected to occur; thus the interpretation of the data in terms of progressive waves is itself oversimplified.

In the second theoretical approach the meanders are assumed steady and controlled predominantly by the process of vortex line stretching as a thin jet which extends to the bottom flows over variable topography. Since the downstream component of flow is quasigeostrophic, the equations of motion are simplified by a transformation of coordinates, i.e. to a local thin curvilinear system coincidental with the axis of the Stream. Let τ denote path length, η be normal to τ, z upward, y northward and x eastward. The equation for the vertical component of vorticity, integrated both across the width of the jet and in the vertical from the sea bottom ($B(x,y) \approx B(\tau)$) to the average sea level (H) is

$$\int_{\text{jet}} \mathrm{d}\tau \left\{ \int_B^H \mathrm{d}z \left\{ \frac{\partial}{\partial \tau}(\kappa V^2) + V\frac{\partial f}{\partial \tau} \right\} + fV(B)\frac{\partial B}{\partial \tau} \right\} = 0, \tag{1}$$

where V is the downstream component, $f = f_0 + \beta y$ the Coriolis parameter and

$$\kappa = (\mathrm{d}^2y/\mathrm{d}x^2)\{1+(\mathrm{d}y/\mathrm{d}x)^2\}^{-\frac{3}{2}},$$

the curvature (assumed constant across the jet). If the current V is known, equation (1) determines the path. If depth variations are small compared to the total depth and $V = V(\eta, z)$ only, equation (1) may be integrated once in τ to yield

$$(\kappa - \kappa_0)\langle V^2 \rangle + f_0 \overline{V}(B - B_0) + \beta \langle V \rangle y = 0, \tag{2}$$

where subscript 0 refers to $(x,y) = (0,0)$, $\langle\ \rangle \equiv \int_{\text{jet}} \mathrm{d}\eta \int_0^H \mathrm{d}z$, $\overline{} \equiv \int_{\text{jet}}$ at $z = B_0$. In geographical coordinates (x,y), equation (2) is a simple ordinary differential equation for the path of the current, which is found from the solution of a spatial initial value problem given the position, direction and curvature of the current at some origin.

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After Warren (1963) integrated equation (2) and found excellent agreement between calculated and observed paths in five specific cases, many believed Gulf Stream meanders to be adequately explained by steady-state topographic theory. However, Niiler & Robinson (1967) and Hansen (1970) have independently shown this conclusion to be incorrect. Hansen's conclusion is based upon a careful term by term analysis of equation (2) for his new data. After an analysis of the general theory of free jets (Robinson & Niiler 1967) Dr Niiler and I performed extensive numerical experiments with a generalized version of equation (2), which was used to relate statistics of the path solution and data to the integral moments $(\langle V\rangle, \langle V^2\rangle, \bar{V})$ of the current

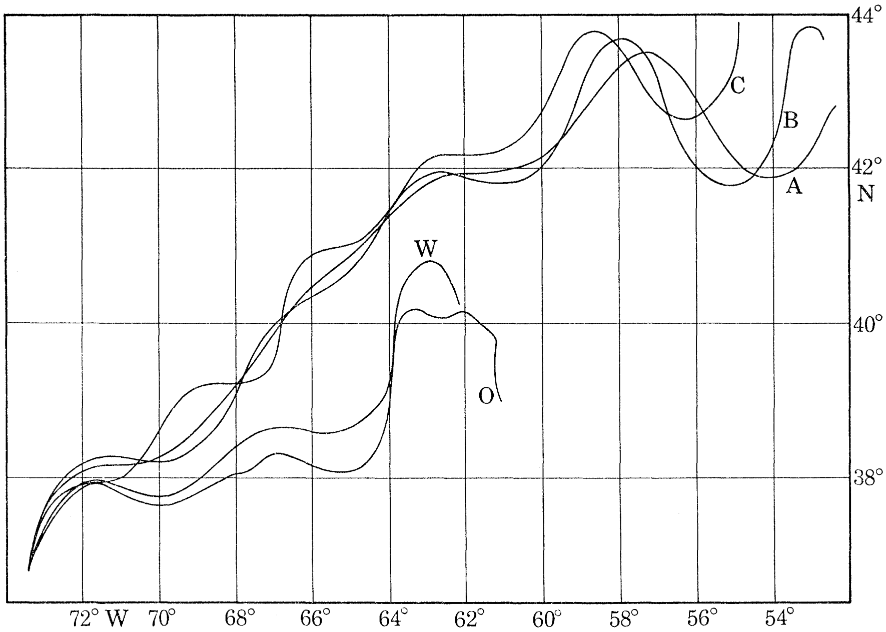

FIGURE 6. Comparison of numerically calculated paths with path observed 8 to 10 June 1950. See Warren (1963, Figure 2). O, Observed Gulf Stream path; W, Warren's calculated path; C, solution for Warren's parameters, bottom averaged 50 km to the right; B, same as C but for an exponential model stream profile (see Niiler & Robinson 1967, § 3); A, same as B but for local rather than averaged bottom.

inferred from hydrographic and velocity measurements. We concluded that the steady-state theory could account for the mean axis and envelope of the paths but not for the wavelength data. If integral moment values are chosen to give correct wavelengths, the current is trapped by the topography and does not descend into the abyssal plain as does nature's Gulf Stream.

The discrepancy between the recent results and the earlier results finds explanation in the approximate method of integration employed by Warren, which was not accurate enough to represent the true solution over the length of path segment which he calculated. Warren treated the bottom as planar over approximately a meander quarter wavelength and patched together the analytical solutions to (2) then available for each step. The maximum step size allowed in our numerical integration was 17 km($\simeq$ 0.05 wavelength); convergence was checked. To illustrate this point in figure 6 is presented the first of Warren's calculated paths, the observed path and the correct numerical solution for Warren's choice of Gulf Stream parameters. Three slightly different solutions are presented for somewhat different assumptions of Stream structure or bottom averaging, in order to demonstrate the insensitivity of the result to such details. Similar results have been obtained for all five of his examples.

The theoretical explanation of the meanders requires a properly posed nonlinear time-dependent theory, in which a process of baroclinic instability is operative and in which vortex-line stretching will undoubtedly play some role in the vorticity balance.

4. The broad current model

The introduction of time dependence into the theoretical considerations introduces a variety of complexities; the thin jet model cannot be simply generalized. For example, the meaning of the Gulf Stream itself as a well-defined current in a time variable flow must be elucidated, and the presence of a transient current must be accompanied by transient motions in the surrounding environment.

To initiate a study of time-dependent topographic meandering, some examples have been studied (Robinson & Gadgil 1970) in terms of the time-dependent generalization of the simplest physical model which exhibits a steady-state topographic meander, namely a barotropic broad current $\partial/\partial y = 0$ flowing over a uniformly sloping sea-bottom, $B-B_0 = Sy(\beta = 0)$. If the current is introduced at the inlet $x = 0$ parallel to the isobaths $U(0,t) = V$ (constant) $V(0,t) = 0$ parallel flow continues downstream (*note* the symbol V is used throughout this paper to represent in every model a flow component associated with the strong downstream component of the Gulf Stream). If, however, a cross-isobath component is maintained at the inlet the downstream flow component remains constant but the cross-stream flow component varies sinusoidally downstream.

The time-dependent generalization is to investigate the flow field $u = V(t)$, $v = v(x,t)$. The cross-stream flow component is governed by the vorticity equation

$$v_{xt} + Vv_{xx} + Sv = 0. \tag{3}$$

The distribution of $v(x,t)$ is obviously analogous to the 'path' of the current; compare the steady version of (3) with the linearized form of (2) with $\beta = 0$ and the same B. We have solved equation (3) for the case of transition from parallel flow to steady-meandering, and for the time variability of the flow induced by slow fluctuations in $V(t)$.

Of special interest is the dependence of the solution of (3) upon general inlet conditions. The equation is a form of the hyperbolic telegraphy equation, but with the usual roles of space and time interchanged. The characteristics are lines of t and $x-Vt$ equal constants for the case of constant V. Thus the inlet $x = 0$ is non-characteristic, and along this line the Cauchy data $v(0,t), V_x(0,t)$ must be specified, i.e. the (spatially) initial values of the direction and curvature of the Stream. If data is specified at some initial instant, along $t = 0$ which is characteristic only, $v(x,0)$ (the initial path of the Stream) is required, not $v_t(x,0)$. Now suppose at one later time (T) a prediction for the path of the Stream through a distance L downstream of the origin is desired (i.e. a solution $v(0<x<L;\,T)$ then the intersection of the characteristics $t = T, x = V(t-T)+L$ with the axes $x = t = 0$ determines the length of time inlet data must be specified (T) and the length of initial path required $(L-VT)$ (Robinson & Gadgil 1970, §3). Clearly if the inlet conditions are monitored for a time L/V they completely determine the desired path.

In order to apply considerations of this kind for the analysis of the meaningfulness of measurements on the actual Gulf Stream, baroclinicity must be taken into account. If the forms $u = V(z)\,\nu(x,t)$ are simply assumed, the baroclinic model vorticity equation integrated over depth assumes a form which can be made identical to (3) provided space and time scales are suitably redefined. If L is chosen as the wavelength of the steady meander, $2\pi\{\bar{V} f_0 S/\langle V^2\rangle H\}^{\frac{1}{2}}$ then the length of time required to measure inlet values of direction and curvature in order to determine the path segment from these quantities alone is $T = 2\pi \langle V\rangle \{H/\langle V^2\rangle \bar{V} f_0 S\}^{\frac{1}{2}}$. Here the bracket indicates integration in depth only and the overbar indicates simply evaluation at $z = 0$, but the model is associated with the thin Gulf Stream by identifying these quantities with the integrals introduced following equation (2) above. Estimates of this time for the Gulf Stream run from a few days up to 2 weeks.

5. An experiment on the time-dependence of the meanders

It is apparent that the testing of theoretical models requires the availability of simultaneous data for a variety of Stream variables (e.g. for the evaluation of $\langle V\rangle$, $\langle V^2\rangle$, $\bar{V}$ as well as the continuous monitoring of the location of a segment of the Stream. The consideration of the above

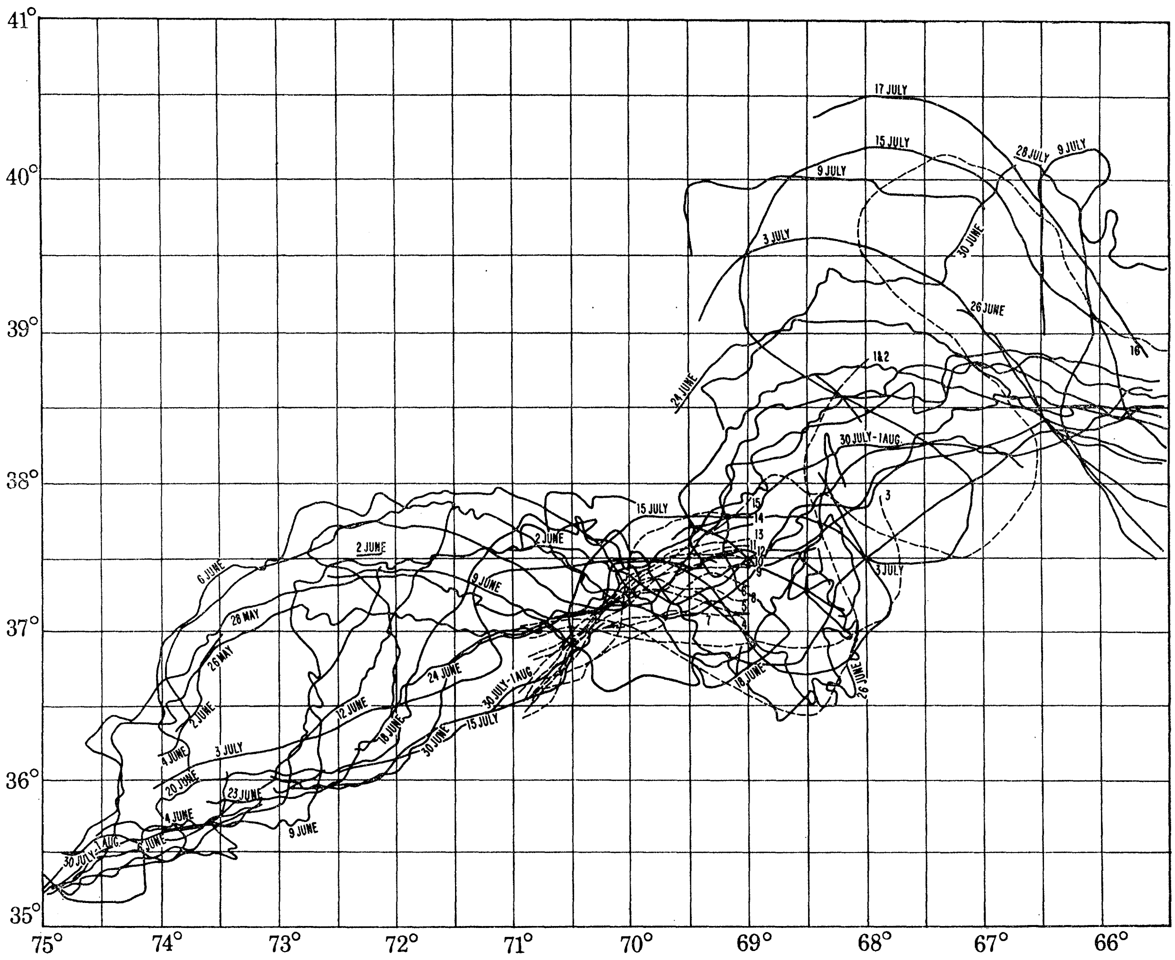

Figure 7. Composite of all paths determined during the experiment in 1969. Solid lines are aircraft paths (ΔT_8). Dashed lines are ship tracks (T_{15}).

paragraph indicated to me that it was feasible for a single ship to collect in a month's cruise sufficient data on the location, direction and curvature of the Stream in the neighbourhood of some longitude to be of considerable usefulness in the definition of the dynamical processes

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underlying the meanders. Consequently an experiment was designed and executed in June and July 1969 which involved the cooperation of oceanographers from Harvard University, the Naval Oceanographic office, and the Woods Hole Oceanographic Institution. The sampling rates were chosen in an attempt to determine unambiguously the time rate of evolution of the large-scale meander features.

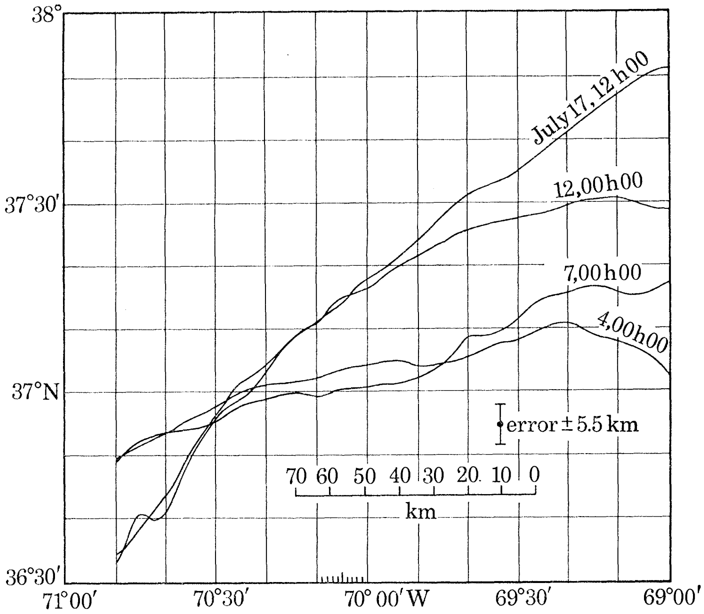

FIGURE 8. Evolution of the Stream path over the duration of the experiment. The paths labels indicate date and time.

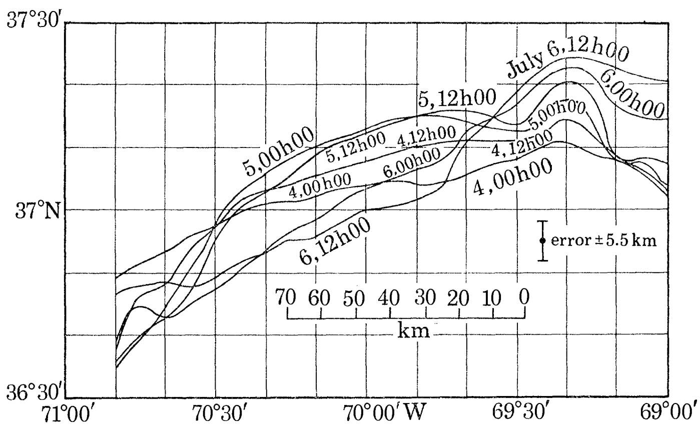

FIGURE 9. Instantaneous paths for midnight and noon 4 to 6 July.

The path data collected consists of: twenty aircraft surface path determinations obtained between 26 May and 1 August from 75° W longitude to as far east as 60° W (Wilkerson & Noble 1970); two relatively long ship tracks of T_{15}, starting on 25 June and extending from 71° W to 68° W and on 17 July from 71° to 64° 31′, and thirteen repetitive determinations of T_{15} between 71° W and 69° W obtained by continuous ship tracking from 3 July to 17 July. In addition four current meters were moored 200 m above the bottom at 70° W and 36° 23′, 36° 43′, 37° 01′, 37° 20′ N on 12 June which were recovered on 13 August (Schmitz, Robinson & Fuglister 1970);

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a deep section was made across the Gulf Stream at 71° to 68° between 30 June and 3 July. Analysis of the data is nearly completed, and a paper is in preparation by myself together with Dr Luyten of Harvard and Mr Fuglister of Woods Hole. Figure 7 displays all of the aircraft and ship tracks obtained.

From the results of the continuous ship tracking of T_{15} over the 2° straddling 70° W we have obtained by interpolation for the first time a succession of 'snapshots' or instantaneous Gulf

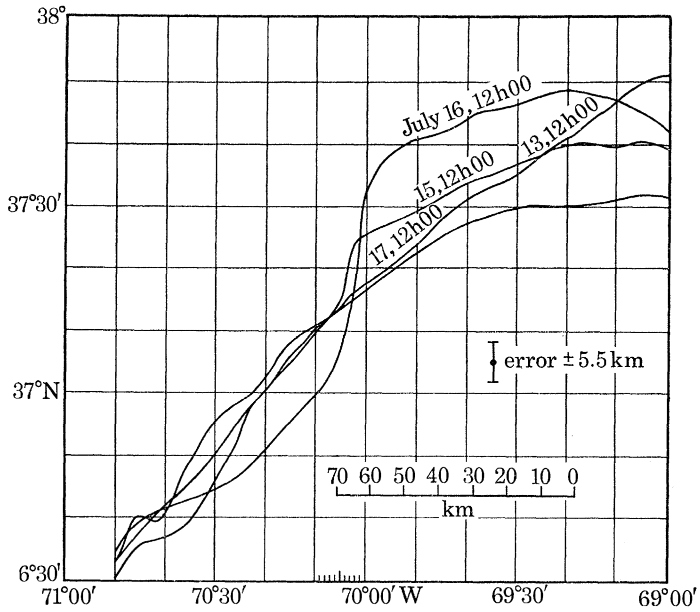

FIGURE 10. Temporary distortion of the Stream on 16 July.

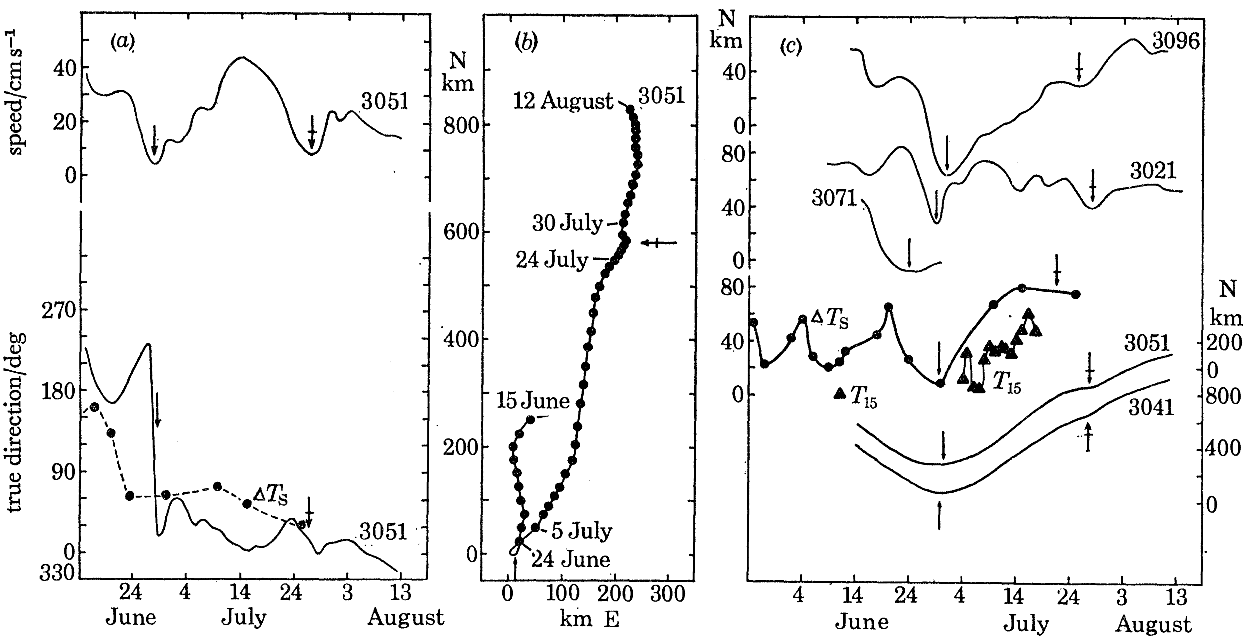

FIGURE 11. (*a*) Speed and direction for current meter 3051 and orientation of the Stream at 70° W as determined from aircraft observations. (*b*) Daily vector virtual displacements. (*c*) Comparison of surface, 200 m, and deep current meter displacements. After Schmitz *et al.* (1970, Figure 1).

Stream paths. The instantaneous paths differ considerably from the successive ship tracks. One cycle took about 26 h, with somewhat more time spent in *XBT* tracking than in streaming. The instantaneous paths were constructed from time plots of T_{15} at selected longitudes. Three results of interest are shown in the accompanying figures. The first (figure 8) shows the slow or secular change in the Stream position and shape over this 2-week period—a result of primary interest. The second (figure 9) reveals the existence in this region for a period of 2 or 3 days of a small-scale disturbance ('wavelength' comparable to the Stream width) which involved displacements or daily fluctuations of the Stream position comparable to those involved in a week of secular movement. The third (figure 10) shows the presence of a large S-shaped feature which passed through the region in little over a day. There is some indication from the aircraft data that this feature was propagating eastward.

Full 2-month records were obtained only from the two southernmost current meters; these records are very similar in both speed and direction (code numbered 3041, the most southern, and 3051). Figure 11*a*, *b* shows the results from the meter located at 36° 43′ N, and figure 11*c* compares a measure of the variability of the Stream as obtained from both these meters with ship's and aircraft surface data. Large speeds, up to 44 cm/s, and speed fluctuations 40 cm/s, with an indicated time scale of about a month, were found. To initiate a comparison of the deep velocity data with the motion of the surface and 200 m temperature data, the former was integrated and the resultant N–S component of displacement plotted against time together with the N–S displacements of T_{15} and ΔT_s on figure 11*c*. The displacement values used were simply those at 70° W. Correlation is apparent, but the meaning is moot. Further interpretation requires a physical model of the transient Gulf Stream. A model of a vertically coherent current which maintains its basic structure while performing time-dependent meanders is presented in the following section.

6. Time-dependent model of a thin coherent jet

Central to the model to be outlined here is the idea that the Gulf Stream essentially maintains its integrity during time-dependent motions on the space and time scales of the meanders. This implies that it is convenient to regard the total Eulerian velocity in the region of the current as composed of the velocity of the Gulf Stream profile (subscript G) and the velocity of the axis of the Stream (subscript A). Thus the model may be loosely thought of as that of a 'wiggly-hose'. The profile velocity vector $\boldsymbol{v}_{\mathrm{G}}$ is compounded of a baroclinic part $\boldsymbol{v}_{\mathrm{T}}$ defined to be zero at the sea bottom, plus a barotropic part, $\boldsymbol{v}_{\mathrm{B}}$, which are assumed always to be in the same direction (the assumption of vertical coherence). The axis velocity $\boldsymbol{v}_{\mathrm{A}}$ is assumed barotropic. The convenience of this decomposition of the total velocity is twofold: it provides a starting-point for the investigation of simplifying dynamical assumptions, and produces a model which contains quantities which can be directly related to presently available or accessible experimental data. The aim is to test the vertical coherency of the Stream, explore the vorticity balance of the meandering, and to develop a scheme for prediction of the path. The model has been developed in cooperation with Dr James Luyten.

Consider the conceptual experiment of a ship at a fixed longitude launching successively surface drogues directly over T_{15} arbitrarily frequently. The downstream locus of the drogues, a function of time coincident with T_{15} and determined by aerial or satellite photography, is defined as the instantaneous axis of the Gulf Stream.

The time-dependent coordinate transformation employed is depicted in figure 12. A geographical point (x, y) within the current is described by the instantaneous normal distance $\eta(x, y, t)$ to the axis, and the longitude X of the axis $y = y(x, t)$ where it is intersected by the normal. The transformation is given implicitly by

$$x = X + \eta \cos\theta(X, t), \quad y = Y(X, t) - \eta \cos\theta(X, t). \tag{4}$$

An arbitrary longitude, $X = X_0$ to the east of Cape Hatteras is to be employed as the westward boundary of the region of the flow to be considered. Since the current is moving back and forth across X_0, as well as fluctuating to the north and south along X_0, an origin point to the west, $X = 0$,

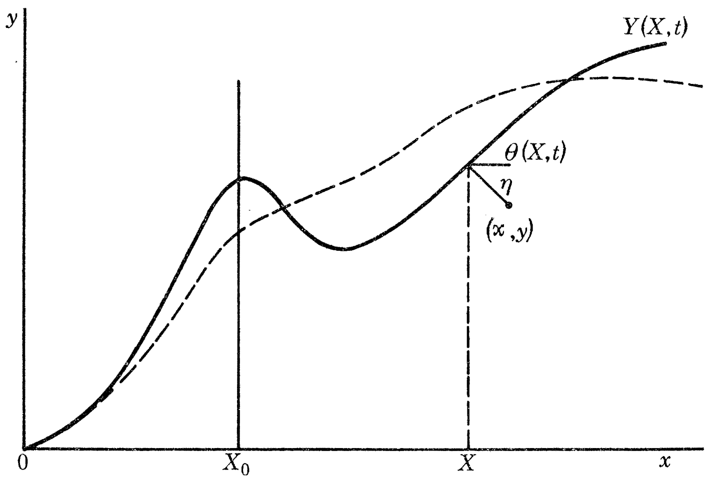

FIGURE 12. The coordinate transformation at time t, solid curve. At a later time the transformation is based on the dotted curve.

is introduced for intermediary argument. The origin is a point at which the current is permanently fixed, e.g. the Straits of Florida. We retain the notation u, v, w for the eastward, northward, upward velocity components and introduce the notation ν, μ for the horizontal components of flow in the instantaneous local normal and tangential direction respectively (i.e. direction defined by a rotation of coordinates through angle $\theta(X, t)$.

To find the axis velocity, first note that by definition a particle on the axis remains on the axis, thus $D(y - Y)/Dt = 0$. Since $\boldsymbol{v} = \boldsymbol{v}_T + \boldsymbol{v}_B + \boldsymbol{v}_A$ but only $\boldsymbol{v}_T$ is z dependent, and under the assumption of vertical coherence the normal component of axis velocity is simply

$$\nu_A = -Y_t \cos\theta. \tag{5a}$$

To find the tangential component, consider a displacement δX, δY during time δt for which $\boldsymbol{v}_G = 0$. The axis is assumed inextensible, and at any instant t a particle at X is at a distance $\tau = \int_0^X \mathrm{d}X'/\cos\theta(X', t)$ from the origin. Then, for this displacement $\tau(X + \delta X, t + \delta t) - \tau(X, t) = 0$, which is an equation for $u_A = \delta X/\delta t$. Since ν_A is known, μ_A can be found and is given by

$$\mu_A = Y_t \sin\theta + \psi(X, t),$$

where
$$\psi \equiv -\int_0^X \mathrm{d}X' Y_{Xt} \sin\theta \equiv -\int_{X_0}^X \mathrm{d}X' Y_{Xt} \sin\theta + \phi(t). \tag{5b}$$

The term $\phi(t)$, a function of time only, represents the rate of slipping back and forth across X_0 due to the amount of the Stream required for the meander pattern westward of X_0. If X_0 is chosen to the west of all eddy-shedding, ϕ will have no long-time average. The structure of the current is of course modified to some extent by the meandering (see, for example, Robinson & Niiler 1967, § 5); the present theory is valid for the case that the meander induced velocity adjustments (subscript M) are of smaller amplitude than the axis velocity components. The total velocity is now further decomposed in that $\boldsymbol{v}_{\mathrm{G}}$ is represented as the sum of the velocity which would be present if there were no meandering plus the meander adjustments. Explicitly, let

$$\mu(x,y,z,t) = \mu_{\mathrm{G}}+\mu_{\mathrm{A}} = V(\eta,z)+\mu_{\mathrm{A}}(X,t)+V_{\mathrm{M}}(\eta,X,z,t), \tag{6a}$$

$$\nu(x,y,z,t) = \nu_{\mathrm{G}}+\nu_{\mathrm{A}} = \nu_{\mathrm{A}}(X,t)+U_{\mathrm{M}}(\eta,X,z,t). \tag{6b}$$

The expressions (6) are to be substituted into the full ideal fluid equation of motion rotated through angle θ and transformed to the coordinates (4). A time-dependent path equation is obtained from the integration of the resultant vorticity equation across the current and in depth in an appropriate approximate form.

The approximation scheme is based upon a scale analysis in the non-dimensional ratios of the scaling amplitudes (denoted by an asterisk) of the various contributions to the velocity, and of the length scales for the current width (l) and meanders (L). We consider the specific problem of meanders induced by fluctuations of inlet angle at longitude X_0. Thus $\theta_t^* = V_{\mathrm{A}}^*/L$ is given as well as the various scales for V, i.e. V_{T}^*, V_{B}^*, $\langle V\rangle^*$, $\langle V^2\rangle^{*\frac{1}{2}}$. The important assumptions are that l/L and V_{T}^*/f_0L, the Rossby number of the basic current based upon the downstream meander scale, are small. It is then found to be necessary that $V_{\mathrm{A}}^* = \langle V^2\rangle^{*\frac{1}{2}}$ and $V_{\mathrm{M}}^* = V_{\mathrm{T}}^{*2}/f_0L$.

The theory is quasigeostrophic. Both $(\boldsymbol{v}_{\mathrm{G}}-\boldsymbol{v}_{\mathrm{M}})$ and $\boldsymbol{v}_{\mathrm{A}}$ are in geostrophic equilibrium, thus $\boldsymbol{v}_{\mathrm{M}}$ provides the first contribution to the divergence. The interaction of the current with the surrounding medium is taken into account by requiring continuity of the normal flow component ν_{A} and the pressure p_{A} at the edges of the current; but the structure of the theory allows this to be done after the path equation is solved and the location of the current is known.

Several contributions to the path equation originate in the downstream momentum equation, namely from the local acceleration of the axis velocity, the curvature term, and the Coriolis acceleration due to the composite rotation $(f+\theta_t)$. The resultant terms appear complicated because of the velocity decomposition. In addition, there is a contribution from the bottom interaction which arises from the integral of the horizontal divergence of $\boldsymbol{v}_{\mathrm{M}}$. The equation for $\beta = 0$, and in non-dimensional form, is given by

$$\cos\theta\frac{\partial}{\partial X}[\nu_{\mathrm{A}t}-2\theta_t(\langle V\rangle+\mu_{\mathrm{A}})-\kappa[\langle V^2\rangle+2\langle V\rangle\psi+\psi^2]-\Sigma f\left\{(\overline{V}+\mu_{\mathrm{A}})\cos\theta B_{\mathrm{X}}+\nu_{\mathrm{A}}\frac{\partial B}{\partial y}(\eta=0)\right\} = 0, \tag{7}$$

where $\Sigma \equiv f_0SL^2/H\langle V^2\rangle^{*\frac{1}{2}}$, S a typical bottom slope (recall figure 12 and equation (5)).

Some physical and mathematical insight is afforded by consideration of the special case of small amplitude meandering over a plane sloping bottom $(B = y)$. Upon linearization (7) then reduces to

$$\langle V^2\rangle Y_{XXX}+2\langle V\rangle Y_{XXt}+Y_{Xtt}+\Sigma\{Y_t+\overline{V}Y_X\}+0. \tag{8}$$

The familiar first term arises from the κV^2 term in the downstream momentum equation, the second term from the Coriolis acceleration $\theta_t V$, the third from local time derivative of the normal component of axis velocity. The fourth term represents the total bottom interaction which is the sum of the usual steady-state contribution from the downstream topographical gradient plus a term due to the influence upon the moving current of the cross-stream gradient.

7. Applications of the model to the experimental data

The model outlined in the preceding section allows the assumption of vertical coherence of the meandering Stream to be tested, by providing a framework within which the near surface data and the bottom data can be interrelated. The T_{15} data provide 2 weeks of continuous information of $Y(X, t)$. For each of the current meter sites all the defined quantities of the model $\eta_i(t)$, $X_i(t)$, $\theta_i(t)$, etc. ($i = 4$, 5 for meters 3041, 3051), can be calculated (we choose $X_0 = 70°$ W). Thus all terms appearing in the axis velocity equations (5a, b) except $\phi(t)$ have been independently

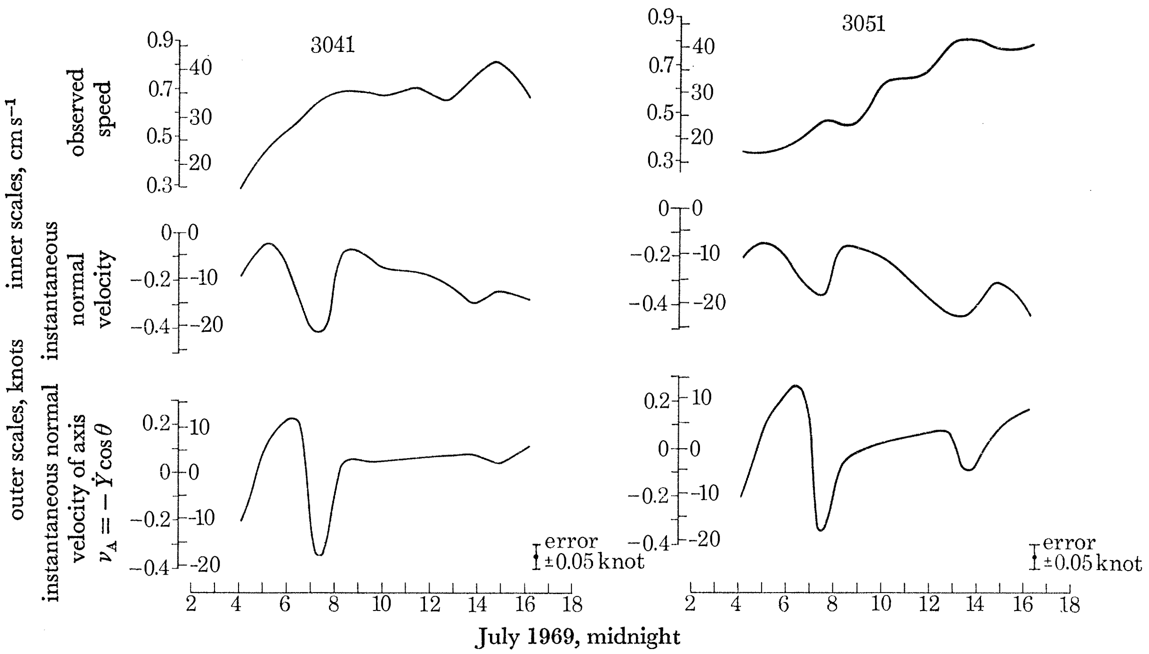

Figure 13. Observed speeds at current meters (upper curves), normal component of the axis velocity as inferred from the deep current meters (centre curves) and from T_{15} (lowest curves).

measured. Now consider equations (6a, b) evaluated at these sites; V_M and U_M are assumed negligible and the left-hand sides are known from the current meter records resolved via $\theta_i(t)$. The normal component equation (6b) allows for a direct test of the vertical coherency assumption. This is shown for each of the current meters in figure 13, which compares $v_i(t)$ and $Y_{ti}\cos\theta(X_i(t), t)$ at each site.

Since the tangential component equations (6a) contain both the Gulf Stream bottom velocity profile $V(\eta, 0)$ and $\phi(t)$, the latter provides a test of the internal consistency of the model. It is unfortunate that the two more northern meters did not yield data which would allow a more complete analysis of the structure of $V(\eta, 0)$ than is presently possible. The present analysis is also restricted by the fact that the time average denoted by (^) of $\phi t + \delta\phi$ over only a 2-week period cannot be expected to vanish. The variation of η_4 is from 50 to 80 km and η_5 from 25 to 55 km. Thus in the vicinity of these sites the bottom profile has been represented by a two-term Taylor

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series expansion about $\eta_0 = 53$ km. Substitution of $V = C_0 + [\eta_i(t) - \eta_0]\, C_1$ into equation (6*a*) yields

$$\mu_i(t) = C_0 + [\eta_i(t) - \eta_0]\, C_1 = Y_{it} \sin \theta_i(X_i(t)) - \int_{X_0}^{X_i(t)} \mathrm{d}X\, Y_{iXt} \sin \theta + \hat{\phi} + \delta\phi(t). \qquad (9)$$

All quantities in two equations (9) have been measured except C_0, C_1, $\hat{\phi}$, $\delta\phi$. The equations are first averaged in time, and then solved for the constants $(C_0 + \hat{\phi}) = 27$ cm/s and $C_1 = 1.8 \times 10^{-6}\,\mathrm{s}^{-1}$. Then each equation can be solved for $\delta\phi(t)$. The balance of terms is similar for both cases and is displayed for $i = 4$ in figure 14*a*. A comparison of $\delta\phi(t)$ values found from the two current meter sites is shown in figure 14*b*.

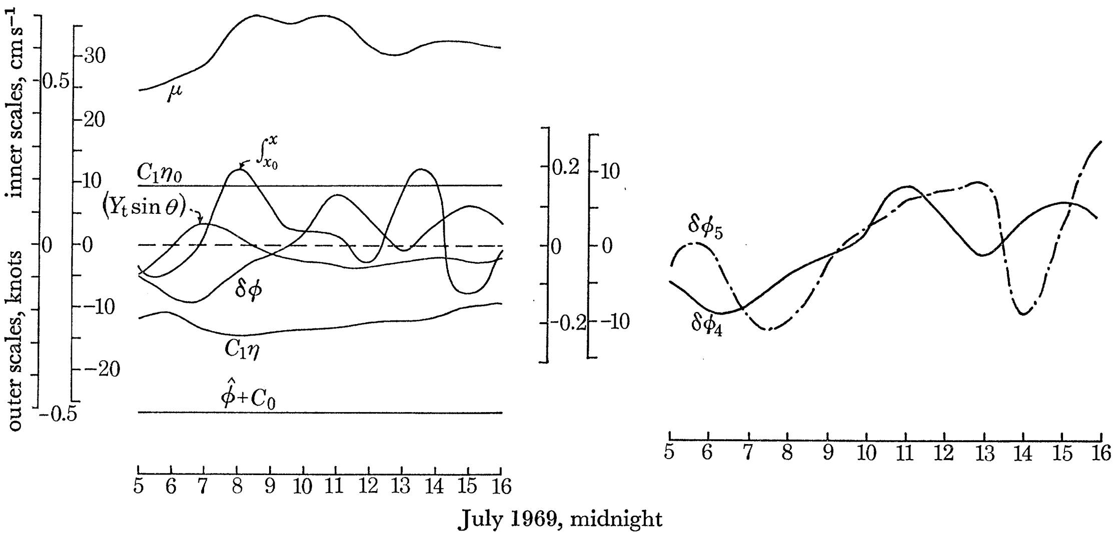

FIGURE 14. The calculation of $\delta\phi$ as inferred from the two current meter records.

Considerable judgement must be exercised in the interpretation of the comparisons exhibited in figures 13 and 14. In my opinion they lend credence to the idea that the meanders extend coherently to the sea bottom, significantly beyond that implied by figure 11. Note that (figure 13) the instantaneous values of the speeds are considerably greater than the magnitudes of the normal components. The agreement in general trend displayed in figure 14*b* is remarkable with respect to the oversimplicity of the model and the quantity and quality of the observations.

An analysis of the implication of the total body of experimental data for the balance of terms in equation (7) has almost been completed. The balance is investigated for the time and space scales of the meanders by averaging over 3 days and 60 km. The most important terms are the familiar curvature term, the total bottom interaction, and the term $\approx \partial\theta/\partial t$. That the phenomenon is not steady-state topographic meandering is strikingly shown by the common occurrence of a situation when not only the magnitude of terms is out of balance, but the signs are opposite to that required.

8. Further theoretical considerations

A complete quantitative understanding to the phenomena of meandering and eddying must ultimately involve a three-dimensional time-dependent numerical model coupled with an extensive observational programme. Numerical analysis and prediction with a model such as that of § 6 may however serve to provide insight into the underlying dynamical processes. We intend to integrate equation (7) not only for real Gulf Stream data but also for a class of simple examples to reveal the range of qualitative behaviour which can be described by the model. For the latter purpose it is instructive to first understand the behaviour of the linearized version, equation (8). Some analysis of (8) exists, because it is identical to an equation which arises in a baroclinic broad current model which has been explored before the derivation of the thin jet model presented above.

The broad current problems have been studied in the context of a rather general model which is quasi-geostrophic in both the cross-stream and downstream directions. The specific type of quasigeostrophy assumed is that pertinent to the Gulf Stream, in which the temperature differences characterizing both the horizontal and vertical directions are of the same order of magnitude; thus the system has effectively less stratification than that characteristic of mid-latitude atmospheric phenomena. In a Rossby number expansion the zero order horizontal velocities and the temperature field are proportional to the derivatives of the zero order pressure $p(x, y, z, t)$ and the zero order vertical velocity vanishes; there is to leading order no contribution from vertical advection in the adiabatic equation (Charney 1962, §3 Case III). The pressure p is governed by an integrodifferential system of equations consisting of the zero order approximation to the adiabatic equation, and the first-order vorticity equation integrated over the depth of the system. Let $\beta = 0$ and the bottom slope uniformly in the y direction. Then the equations are

$$P_{zt} + J(p, p_z) = 0, \tag{10}$$

$$\int_0^1 \mathrm{d}z[\nabla_{\mathrm{H}}^2 p_t + J(\nabla_{\mathrm{H}}^2 p, p)] + \Sigma p_x(x, y, 0, t) = 0, \tag{11}$$

in non-dimensional form, where $\nabla_{\mathrm{H}}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and Σ is bottom interaction parameter similar to that defined following equation (7). It is instructive to consider the problems of 'structure' and 'path' of a meandering current in terms of this model. The path equations discussed above are clearly analogous to the integrated vorticity equation (11).

We consider here a proper baroclinic version of the broad current model introduced in §4. Let $\partial/\partial y = 0$, the downstream component at the inlet be $V(z)$, and the cross-isobath y component of flow be $v(x, z, t)$. Then equations (10) and (11) and the geostrophic relationships lead to

$$v_{zt} + V(z)\, v_{xz} - V_z v_x = 0, \tag{12}$$

$$\int_0^1 \mathrm{d}z[v_{xt} + V v_{xx}]\,\mathrm{d}z + \Sigma v(x, 0, t) = 0 \tag{13}$$

(Robinson & Gadgil 1970). The separated form $v(x, z, t) = [V(z) - \sigma/k] \exp \mathrm{i}(kx - \sigma t)$ solves (12), and when inserted into (13) yields the dispersion relationship

$$-k\sigma^2 + 2\langle V\rangle k^2\sigma - \langle V^2\rangle \kappa^3 + \Sigma(-\sigma + \bar{V}k) = 0. \tag{14}$$

Here again the bracket signifies integration in depth only and the overbar evaluation at $z = 0$. Equation (14) is identical to the dispersion relationship resulting from the insertion of the form $Y \approx \exp\{\mathrm{i}(kx - \sigma t)\}$ into the linearized path equation (8) for the thin jet.

We consider here for simplicity the flat-bottom case of $\Sigma = 0$. For the spatial instability problem, σ given and real, (14) yields

$$k = (\sigma/\langle V^2\rangle)\,[\langle V\rangle \pm \surd(\langle V\rangle^2 - \langle V^2\rangle)]. \tag{15}$$

Thus a baroclinic current is necessarily unstable in the sense that oscillations at the inlet will always amplify exponentially downstream. This is the spatial version of the temporal instability model introduced by Tareev (1965) for application to the Gulf Stream.

9. An eddy-production mechanism

In § 4 a discussion was presented of the dependence upon general inlet conditions of solutions to the hyperbolic equation (3), which is the barotropic limit of equation (13) and an 'equivalent baroclinic' interpretation was suggested. A proper study of the baroclinic problem has been initiated by Sulochana Gadgil (1970) in her doctoral dissertation. In principle, the general solution to (12) determines the vertical structure of the flow and contains arbitrary functional dependence upon (x, t) which is then determined by the partial differential equation in (x, t) obtained by inserting the solution into (13) and performing the z integration explicitly.

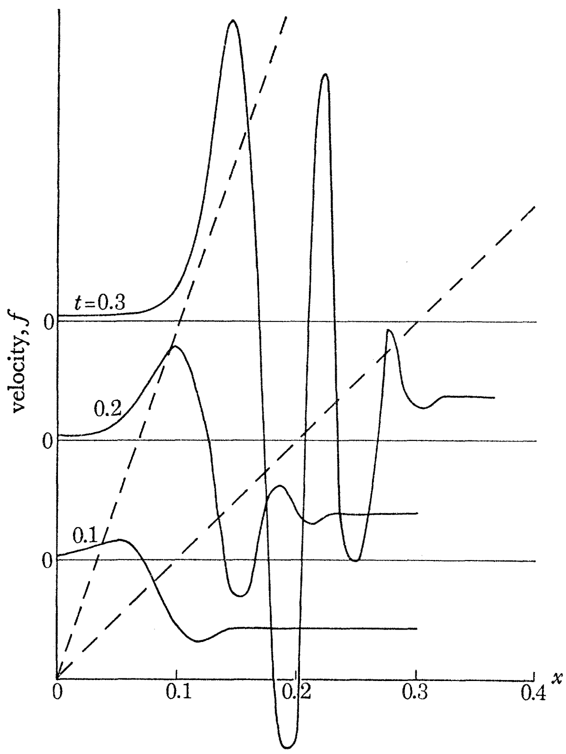

Figure 15. The cross-isobath bottom velocity f against x at three successive times. After Gadgil (1970, Figure 4.6).

To determine the solution $v(x, z, t)$ of equation (12) requires the specifications of, (i) the velocity field at the initial instant $v(x, z, 0)$, (ii) the time evolution of the velocity at the inlet, $v(0, z_0, t)$. The adiabatic equation (12) essentially governs the evolution of the baroclinic component of the flow and the specification of the barotropic component. The barotropic component is not however free; its evolution is determined by the depth averaged vorticity equation. Recall the level of no motion problem of classical oceanography. The subsequent analysis bears interestingly upon the reference level problem in geostrophic calculations, since knowledge of $v(x, z_0, t)$ is replaced by information at the inlet.

The reference level z_0 is of course arbitrary and without loss of generality we take $z_0 = 0$, $v(x, 0, t) \equiv f(x, t)$. The domain of interest is $x > 0$; equation (12) has been solved in general by Laplace transformation of the x-dependence and the subsequent use of Riemann's method. The solution at a point (x, z, t) does not of course depend upon all the initial and inlet data. The solution is then substituted into $(\partial/\partial t+\bar{V}\partial/\partial x)$ (13) and the vertical integrals performed. The resultant equation

$$f_{xtt}+2\langle V\rangle f_{xxt}+\langle V^2\rangle f_{xxx}+\Sigma(f_t+\bar{V}f_x) = 0 \qquad (16)$$

is identical to the linearized free jet path equation (8).

The equation for the cross-isobath barotropic flow component, or path equation (16) is elliptic because of the baroclinicity of the flow, i.e. since $\langle V^2\rangle > \langle V\rangle^2$. The ellipticity of the operator is of course related physically to the instability found in equation (15).

The usual well-posed Neumann and Dirichlet problems for an elliptic operator in which data is specified on a closed curve in the x-t plane are not physically the correct ones here, since the solution would depend upon future data. As for equation (3) the Cauchy problem is the one of interest. Again at the inlet $x = 0$ the direction $f(0,t)$ and curvature $f_x(0,t)$ are required; $f_{xx}(0,t)$ is known via an auxiliary equation.

As a simple example consider the case of a flow over a flat bottom ($\Sigma = 0$) that has initially no cross-isobath component; then, at $t = 0$, at the inlet the cross-isobath component builds up continuously and is maintained at a constant value. The change is made in a vertically coherent fashion. The solution, which is illustrated in figure 15, exhibits a localization of the instability phenomenon which is very suggestive of eddy-formation. Recall figure 5. The region in which the amplitude becomes arbitrarily large sufficiently far from the inlet is confined to a triangular region of the (x,t)-plane, between the dashed lines $x = \langle V^2\rangle t/\{V\pm\sqrt{(\langle V^2\rangle-\langle V\rangle^2)}\}$. In this region oscillations occur which decrease in wavelength as they amplify. The appearance of this phenomena in solutions of the related nonlinear equation (7) will I believe, produce eddies.

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